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Math 451: Intro. to General Topology — HOMEWORK 4 — **Due: Mar 10, 23:59**

Definition. In a topological space X, we say that a sequence $(x_n) \subseteq X$ **converges** to $x \in X$ if for each open $U \ni x$, we have $\forall^\infty n \; x_n \in U$. In this case, we also say that x is a **limit** of (x_n).

1. Prove that in a Hausdorff topological space, a sequence has at most one limit.

2. Prove that $\mathbb{Q}$ with its usual topology induced by the metric $d(x,y) := |x-y|$ is zero-dimensional, i.e. it has a basis consisting of clopen sets.

3. Let $\mathcal{E}$ be the collection of all arithmetic progressions in $\mathbb{Z}$ (i.e. cosets of nontrivial subgroups of $\mathbb{Z}$), and recall that the profinite topology $\mathcal{T}_p$ on $\mathbb{Z}$ is the one generated by $\mathcal{E}$.

(a) Prove that $\mathcal{E}$ is closed under nonempty finite intersections, i.e. if $A, B \in \mathcal{E}$ have a nonempty intersection, then $A \cap B \in \mathcal{E}$. Thus, $\mathcal{E}$ is a basis for $\mathcal{T}_p$.

(b) Observe that this topology is **perfect**, i.e. no singleton is open.

(c) For $z \in \mathbb{Z}$, let $\|z\| := \sum_{k \nmid z, k \geq 1} 2^{-k}$, where $k \mid z$ means k divides z, so the sum is over all $k \geq 1$ which do not divide z. Prove that this satisfies the triangle inequality $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathbb{Z}$ (and hence $\|\cdot\|$ is a norm). Deduce that $d(x,y) := \|x-y\|$ is a metric on $\mathbb{Z}$.

(d) Prove that this metric d induces the profinite topology on $\mathbb{Z}$.

HINT: It is enough to show that every open ball wrt d is an open set in $\mathcal{T}_p$, and conversely, every arithmetic progression in $\mathcal{E}$ is an open set wrt d.

REMARK: Thus, $\mathbb{Z}$ with this topology is metrizable, zero-dimensional, perfect, and countable. Hence, by Sierpiński's theorem, $\mathbb{Z}$ with the profinite topology is homeomorphic to $\mathbb{Q}$ (with the subspace topology inherited from $\mathbb{R}$).

4. In $\mathbb{R}^d$ the usual (Euclidean) topology, prove that the open boxes $\prod_{i=1}^d (a_i, b_i)$ with rational coordinates $a_i, b_i \in \mathbb{Q}$ form a basis.

5. Fill in the details in the argument below to prove that in a second countable topological space X, every basis $\mathcal{B}$ admits a countable subset $\mathcal{C} \subseteq \mathcal{B}$, which is still a basis.

(i) Fix a countable basis $\mathcal{U}$ and note that each $U \in \mathcal{U}$ is itself a second countable space (with subspace topology), hence Lindelöf.

(ii) Therefore, each $U \in \mathcal{U}$ is a *countable* union of sets in $\mathcal{B}$, i.e. $U = \bigcup \mathcal{B}_U$ for some countable collection $\mathcal{B}_U \subseteq \mathcal{B}$ (chosen for each U using AC).

(iii) Then $\mathcal{C} := \bigcup_{U \in \mathcal{U}} \mathcal{B}_U$ is a countable basis.

6. Prove that the Lindelöf property is inherited by closed subspaces, i.e. if X is a Lindelöf topological space and $Y \subseteq X$ is closed, then Y is also Lindelöf (in the subspace topology).

7. Let (X,d) be a Lindelöf metric space and prove that X is separable.

HINT: For each $\frac{1}{n}$, consider an appropriate open cover of X.

8. Let $(\mathbb{R},\mathcal{S})$ be the Sorgenfrey line, i.e. $\mathcal{S}$ is the topology generated by the collection $\mathcal{B}$ of half-open intervals $[a,b)$, where $a,b \in \mathbb{R}$, $a < b$.

(a) Verify that $\mathcal{B}$ is a basis. (I said this in class but I am not sure if everyone followed why, so I want to be sure.)

(b) Show that the Sorgenfrey topology is separable and first countable.

(c) Prove that the Sorgenfrey topology is not second countable.

HINT: Let $\mathcal{C} \subseteq \mathcal{B}$ be a basis and show that for each $a \in \mathbb{R}$, there has to be a set in $\mathcal{C}$ of the form $[a,b)$ for some $b > a$.

9. [*Optional*] Fill in the details in the argument below to prove that the Sorgenfrey line $(\mathbb{R},\mathcal{S})$ is Lindelöf.

(i) Let $\mathcal{U} \subseteq \mathcal{S}$ be an open cover of $\mathbb{R}$. Call a set $A \subseteq \mathbb{R}$ **countably coverable** if there is a countable $\mathcal{U}_A \subseteq \mathcal{U}$ covering A, i.e. $A \subseteq \bigcup \mathcal{U}_A$. Fix an arbitrary $a \in \mathbb{R}$ and define

$$X_a := \{b \in [a,\infty) : [a,b] \text{ is countably coverable}\}.$$

Prove that X_a is unbounded.

HINT: Suppose towards a contradiction that $b := \sup X_a < \infty$ and prove that $[a,b+\varepsilon]$ must be countably coverable for some $\varepsilon > 0$.

(ii) Deduce that every interval $[a,b]$, where $a < b, a, b \in \mathbb{R}$, is countably coverable.

(iii) Conclude that $\mathbb{R}$ is countably coverable.

10. Let X be a topological space, $x \in X$, and $\mathcal{B}_x$ be a neighbourhood basis at x. Let $(x_n) \subseteq X$ and show that $x_n \to x$ if and only if for each $U \in \mathcal{B}_x$, we have $\forall^\infty n$, $x_n \in U$.